LAGRANGE INTERPOLATION FORMULA WITH REMAINDER

Idea: We are in the situation where the values of a function are known at some finite number of nodes. Then we can find a polynomial that interpolates the function at these nodes. If the function is smooth enough we can estimate the error of the approximation.

Theorem 0.1. *Let f be differentiable $n+1$ times on $[a,b]$. Consider the interpolating polynomial $p(x)$ of f at $n+1$ nodes $x_0, x_1, ..., x_n \in [a,b]$. Explicitly, $p(x) = \sum_{j=0}^{n} f(x_j)L_j(x)$ where $L_j(x) = \prod_{k=0,k\neq j}^{n} \frac{x-x_k}{x_j-x_k}$.*

Then for any $x \in [a,b]$, $f(x) = p(x) + R(x)$ where

$$R(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{k=0}^{n} (x - x_k) \tag{0.1}$$

for some $\xi = \xi(x) \in [a,b]$.

Proof: Consider x fixed. Then we are going to find $\xi \in [a,b]$ which makes (0.1) hold. Define the number E by the equation $f(x) = p(x) + E\prod_{k=0}^{n}(x-x_k)$. We want to show that $E = \frac{f^{(n+1)}(\xi)}{(n+1)!}$ for some ξ. This is tricky: let $g(t) = f(t) - p(t) - E\prod_{k=0}^{n}(t-x_k)$. A direct differentiation shows that $g^{(n+1)}(t) = f^{(n+1)}(t) - E(n+1)!$. Then it is enough to prove that $g^{(n+1)}(\xi) = 0$ for some $\xi \in [a,b]$. This is done by noting that $g(x) = 0$ and $g(x_k) = 0$ for $k = 0, 1, ...n$. If x coincides with an x_k, the whole proof is trivial. Otherwise, g has at least $n+2$ distinct zeros on $[a,b]$. Then $g^{(1)}$ has at least $n+1$ distinct zeros on $[a,b]$ because g is differentiable on $[a,b]$.(application of Rolle's theorem). Continue this way to conclude that $g^{(n+1)}$ has a zero $\xi \in [a,b]$. □