Recap last time

- Derived Lorentz transformation which tells you how position and time coordinates are related in different (inertial) FOR
- Form required only that $c^2 t^2 - x^2$ same in all FOR. This is the mathematical expression of the principle of relativity ...
- Interpret this as a distance in space-time
Transformation between coordinates

- Taking dot products of first eqn. with respect to $\mathbf{i}'$ leads to

$$x' = x (\mathbf{i}.\mathbf{i}') + y (\mathbf{j}.\mathbf{i}')$$

But $\mathbf{i}.\mathbf{i}' = \cos (\theta)$ and $\mathbf{j}.\mathbf{i}' = \sin (\theta)$ so that the transformation of the coordinates may be written

$$x' = \cos (\theta)x + \sin (\theta)y$$
$$y' = -\sin (\theta)x + \cos (\theta)y$$

- Notice that the length of the position vector $x^2 + y^2 = (x')^2 + (y')^2$ is **invariant** with respect to transformation between different (rotated) coordinate systems (although the component values $x$ and $y$ are certainly not).
Definition of vectors

- Can define 2d (space) vector as set of 2 numbers \((x, y)\) which transform according to this rule as coordinate system is rotated.
- This rule is determined by requiring that the length \(x^2 + y^2\) same in all frames ...
- Sound familiar ?
- Natural way to map the idea of an invariant spacetime interval under LT into the language of 2d rotations ...
The correspondence

- (Minus) spacetime distance squared $x^2 - c^2 t^2 = x^2 + T^2$ if $T = i c t$.
- Consider previous construction with $y \rightarrow T$.
- Set $x' = 0$. Rotation angle $\tan \theta = x/T$
- Thus $\tan \theta = -i \frac{v}{c}$. Using $\frac{1}{1 + \tan^2(\theta)} = \cos^2(\theta)$ find

\[
\cos(\theta) = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \gamma
\]

\[
\sin(\theta) = \tan(\theta) \cos(\theta) = \gamma \frac{v}{c}
\]

- Have re-derived Lorentz transformation as rotations in complex plane!!
Spacetime vectors

- Define spacetime vector (sometimes called 4-vector) as a vector that transforms same way as the spacetime coordinate vector \((x, ct)\).
- Has 3 space and 1 time component. \((a_t, a_x, a_y, a_z)\).
- Length \(a_t^2 - a_x^2 - a_y^2 - a_z^2\) same in all inertial FOR.
- Components transform using LT.
- Most important: If theory formulated in terms of such vectors we are guaranteed that equations will look same in all inertial FOR!
Lorentz transformation as a matrix ...

- First notation. Write a space-time vector $v_\mu = (v_t, v_x, v_y, v_z)$. The index $\mu$ runs from 1 to 4 (not 1 to 3 as for regular vectors).

- Lorentz transformations look like matrices acting on these vectors e.g. (restrict to just $x, t$)

\[
\begin{pmatrix}
    x' \\
    ct'
\end{pmatrix} = \gamma \begin{pmatrix}
    1 & \beta \\
    \beta & 1
\end{pmatrix} \begin{pmatrix}
    x \\
    ct
\end{pmatrix}
\]

$\beta = v/c$

Shorthand:

\[
x'_{\mu} = \sum_{\nu} L_{\mu\nu} x_{\nu}
\]
Consider 2 such space-time vectors eg. $a_\mu = a$ and $b_\mu = b$. Consider an equation that looks like in some FOR

$$a_\mu = b_\mu \quad \text{or} \quad a = b$$

What does this look like in another FOR? Act with $L$ on both sides

$$La = Lb$$

But $a' = La$ etc so this reads

$$a' = b'$$

Same form in different FOR!! Equations *automatically* respect relativity if written in terms of space-time (4) vectors!
Consider worldline of particle in spacetime.

Any small portion of it looks like straight line with components (in some FOR) \((c\Delta t, \Delta x, \Delta y, \Delta z)\)

This is simplest example of spacetime vector.

Points along worldline at that point.

Calculate proper time for this small displacement \(\Delta \tau\) - spacetime invariant (scalar)

Then from previous vector can construct another vector

\[
P = m_0 \left( \frac{c\Delta t}{\Delta \tau}, \frac{\Delta x}{\Delta \tau}, \frac{\Delta y}{\Delta \tau}, \frac{\Delta z}{\Delta \tau} \right)
\]

\(m_0\) some constant.
Why?

- What is $m_0$? What is the physical interpretation of this vector?

- $\Delta \tau^2 = \Delta t^2 (1 - \frac{1}{c^2} \frac{\Delta x^2}{\Delta t^2})$.

- Consider $v = \frac{\Delta x}{\Delta t} \ll c$. Spatial components of $P$ eg. $P_x = m_0 \frac{dx}{d\tau}$ look like velocity times $m_0$. Thus treat $m_0$ as form of mass - “rest mass”.

- Thus spatial components of $P$ are relativistic generalization of momentum!

- But what about the time component $m_0 c \frac{dt}{d\tau}$?
Consider small $v$. Taylor expand the square root

$$m_0 c \frac{dt}{d\tau} = m_0 \gamma c = m_0 c \left(1 + \frac{1}{2} \frac{v^2}{c^2} + \ldots\right)$$

Contains the Newtonian kinetic energy $K/c$ plus a constant.

Thus it is a relativistic generalization of energy of motion! $P_t = E/c$

But notice it has a value even when at rest $m_0 c$! i.e there is a rest energy $E = m_0 c^2$ But since $c$ is a constant shows that this rest energy is just a measure of mass.

Famous equivalence of mass and energy. Large amount of energy from small mass ...
Rederived LT by analogy with 2d spatial rotations. Concept of spacetime vector.

Laws of physics should be written in terms of such vectors. **Automatically ensures principle of relativity**

Simplest example of such a vector — energy-momentum vector

\[ P = \left( \frac{E}{c}, p_x, p_y, p_z \right). \]

Energy becomes time component of 4 vector. Momentum is space part. But both of these are modified from Newtonian expressions as \( v \rightarrow c \)

Furthermore, time component reveals equivalence of mass and energy. Energy in rest frame equals (invariant) mass \( m_0 \)
More on Energy-momentum vector

- Special relativity unites time with space but also energy with momentum and mass with energy!
- $E/c = m_0 c \gamma$. Hence $E \to \infty$ as $v \to c$. Physical reason why cannot go faster than speed of light.
- $p = m_0 v \gamma$. Momentum also infinite as $v \to c$. Notice relativistic momentum exceeds Newtonian expression since $\gamma \geq 1$.
- What is mass? Energy measured in FOR in which particle is at rest ...
- Length of vector: $E^2/c^2 - p_x^2 - p_y^2 - p_z^2 = m_0^2 c^2$
Examples

▶ The Large Hadron Collider is the largest laboratory in the world for doing high energy physics. In it protons are accelerated to speeds close to the speed of light and made to collide. The protons in the beam have energies of 7000 Gev (1 Gev is $1.6 \times 10^{-10}$ Joules). The proton rest mass is $1.67 \times 10^{-27}$ kgs. What is the ratio of the speed of the particles to the speed of light? (take $c = 3.0 \times 10^8 m/s$).

▶ The protons and antiprotons are accelerated around a circular ring of circumference approx 27 kilometers. How long (in the rest frame of the accelerator) does it take a proton/antiproton to go once around the ring? How long does one of the protons think it takes..?
Many particles

- For system of particles get the total energy-momentum vector by adding up the energies and momenta for each.

\[ P_{\text{tot}} = \left( \sum_i E_i/c, \sum_i p_i \right) \]

- What is mass of system? Length of total energy momentum vector. i.e \( \frac{1}{c} \sqrt{E_{\text{tot}}^2/c^2 - p_{\text{tot}}^2} \) In general this is not the sum of all rest masses!

- i.e the total length of a vector is not in general the sum of a single component of each vector!

- This is the price one pays for unifying mass energy with momentum...
Consider 2 particles both of (rest) mass $m$, one at rest the other moving with total energy $4mc^2$. What is

1. Total energy of the system
2. Total momentum of the system
3. Total mass of the system

ans: $5mc^2$, $\sqrt{15}mc$, $\sqrt{10}m$
Photons are massless particles. Hence
\[ m^2c^2 = 0 = E^2/c^2 - p^2. \text{ i.e } E = pc. \]
Consider two photons - initially travelling in same direction with energies \( E = 3c \) and \( E = 1c \).

1. What is their total energy ?
2. Hence what is their total momentum ?
3. What is the mass of the 2 photons ?
4. What happens when photons headed in opposite directions ?

ans: 4c, 4, 0, \( \sqrt{12}/c \)
Mass has different interpretation in relativity than Newtonian theory. Mass becomes rest mass – length of particles energy-momentum vector.

Total mass of system is just length of total energy-momentum vector of system.

In rest frame of particle proportional to energy via $E = m_0 c^2$.

Energy and momentum are united into more fundamental object – energy-momentum vector $\mathcal{P}$. 
In Newtonian mechanics $E$ and $p$ conserved separately. In relativity simply have conservation of $P$. Valid in any inertial FOR.

Yields conservation of momentum and energy at same time ...

Length of $P$ conserved – conservation of (total) mass also!
Why should $\mathcal{P}$ be conserved?

Actually this is implied by principle that proper time is maximum for free particles....
Draw a spacetime diagram in some FOR. Consider the worldline of a particle traveling between two points A and B with coordinates \((0, 0)\) and \((X, T)\).

Draw an intermediate position P with coordinates \((x, t)\).

Proper time for particle to go from A to B is sum of two parts
\[
\tau = \tau_{AP} + \tau_{PB}.
\]

Principle of maximal ageing tells me that P must be chosen so that
\[
\frac{d\tau}{dt} = 0 \quad \text{and} \quad \frac{d\tau}{dx} = 0
\]
Thus

\[ \frac{1}{2\tau_{AP}} \frac{d\tau_{AP}^2}{dt} + \frac{1}{2\tau_{PB}} \frac{d\tau_{PB}^2}{dt} = 0 \]

Using \( \tau_{AP}^2 = t^2 - x^2/c^2 \) and \( \tau_{PB}^2 = (T - t)^2 - (X - x)^2/c^2 \)
we find

\[ \frac{t}{\tau_{AP}} = \frac{T - t}{\tau_{BP}} \]

Putting \( t \to \Delta t, \tau_{AP} \to \Delta \tau \) find \( \frac{\Delta t}{\Delta \tau} \) is conserved along classical path (path of maximal ageing)

Similarly can vary \( x \) and find \( \frac{1}{c^2} \frac{\Delta x}{\Delta \tau} \) is conserved also ....

Thus maximal ageing implies energy and momentum are conserved!
A photon moving with energy $E$ collides with a stationary atom with (rest) mass $m$. The photon is absorbed and the atom recoils. Work out formulae for:

- The mass of the atom after collision
- The momentum of the atom after the collision.
- How fast it is traveling after collision (as viewed from the original FOR at which it was at rest)

A typical visible light photon carries approx $1 \times 10^{-18}$ J of energy while a hydrogen atom has mass approx $10^{-26}$ kg. What is the velocity of the recoiling H atom?
Conservation of energy and momentum:

\[ E + mc^2 = E_f \]
\[ p = E/c = p_f \]

Total mass \( M^2 c^2 = E_f^2/c^2 - p_f^2 = 2Em + m^2 c^2 \)

To find velocity use \( \gamma = E_f/mc^2 \). Find:

\[ \frac{\nu}{c} = \frac{1}{1 + mc^2/E} \]

Putting \( E = 10^{-18}, m = 10^{-26}, c = 3 \times 10^8 \) find \( \frac{\nu}{c} \sim 10^{-8} \) ie few m/s.
Made fast paced but somewhat thorough trek through the most important pieces of Special Relativity: all inertial FOR equally good for discovering/using laws of physics

Key concept: time and space have no absolute meaning - depend on FOR. But combine as elements of spacetime - find absolute concepts eg distance in spacetime.

Relativistic physics written in the language of spacetime (4-) vectors.

Consequences dramatic: time dilation, equivalence of energy and momentum, rest mass energy $E = mc^2$, ....

But problems still remain ... led to General Relativity